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Thermosolutal convection with finite amplitude in a rotating fluid under the effect of magnetic field

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Abstract In this paper, the effect of magnetic field on the convective pattern in a horizontal layer of fluid under rotation is examined for both stationary and oscillatory case. Existence of overstability is ensured for the problem for particular set of parameters influencing the motion *viz* Q–the rotation parameter, η'–the parameter responsible for magnetic effect, τ the ratio of solutal and temperature diffusivity *etc*

Keywords Thermosolutal convection, fluid under rotation, overstability

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1. Introduction

Thermosolutal convection was first considered by Veronis [1] subjected to a salinity gradient produced at the interface of a layer of fluid heated from below. Sengupta and Gupta [2] generalized the problem by taking into account the rotational effect of the system. It was also pointed out by them that for disturbances of finite amplitude subcritical instability may occur revealing the fact that the system becomes unstable to steady finite amplitude disturbances before the system becomes unstable to infinitesimal disturbances. This work yields a conclusion that due to the magnetic influence the critical Rayleigh decreases giving rise to the fact that the system has a destabilizing effect for the magnetic field. Further occurrence of subcritical instability is possible in this situation also for finite amplitude disturbances

2. Mathematical formulation

Taking z-axis as vertical, a horizontal layer of fluid heated and salted from below is considered. The layer is under the effect of rotation with angular velocity Ω about z-axis and magnetic field is influencing it. The two bounding surfaces are taken as free and perfect conductors of heat and salt. For the sake of convenience, two dimensional motion is considered. Following Sengupta and Gupta [2] and Chandrasekhar [3] under Boussinesque approximation, the guiding equations are

$$\frac{\partial \boldsymbol{q}}{\partial t} + \boldsymbol{q}\cdot\nabla\boldsymbol{q} = -\frac{1}{\rho_m}\nabla p' - 2\boldsymbol{\Omega}\times\boldsymbol{q} + g(\alpha T - \beta S)\boldsymbol{k} + \nu\nabla^2\boldsymbol{q} + Q_1\nabla(\nabla^2 h), \tag{1}$$

where $\boldsymbol{q} = (u, v, w)$, $\boldsymbol{h} = (0, 0, h_z)$

$$\frac{\partial T}{\partial t} + \boldsymbol{q}\cdot\nabla T - w\frac{\nabla T}{d} = k_T\nabla^2 T \tag{2}$$

$$\frac{\partial S}{\partial t} + \boldsymbol{q}\cdot\nabla S - w\frac{\nabla S}{d} = k_S\nabla^2 S \tag{3}$$

$$\frac{\partial h_z}{\partial t} + \boldsymbol{q}\cdot\nabla h_z = \eta\nabla^2 h_z + H_z\frac{\partial w}{\partial z} \tag{4}$$

$$\nabla\cdot\boldsymbol{q} = 0 \tag{5}$$

where $\rho = \rho_m[1 - \alpha T - \beta S]$. (6)

To make the equations dimensionless, we introduce the dimensionless variables starred as,

$$\boldsymbol{q} = \frac{k_T}{d}\boldsymbol{q}^*,\ t = \frac{d^2}{k_T}t^*,\ (x, y, z) = d(x^*, y^*, z^*),$$
$$T = \Delta T T^*,\ S = \Delta S S^*,\ \omega = \frac{\rho^* d^2}{\rho_m \nu k_T},\ \Psi = k_T\Psi^*,$$
$$h_z = \Delta h_z k^* \tag{7}$$

and the stream function Ψ as

$$u = \frac{\partial\Psi}{\partial z} \text{ and } w = -\frac{\partial\Psi}{\partial x}, \tag{8}$$

so that the equation of continuity (5) is satisfied.

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Utilising (7) and (8) in (1–4) and eliminating ω, after dropping star, we get

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t}-\nabla^2\right)\nabla^2\Psi=-R\frac{\partial T}{\partial x}+R_S\frac{\partial S}{\partial x}+Y\frac{\partial v}{\partial z}+\frac{1}{\sigma}J(\Psi,\nabla^2\Psi)-Q_1^*\frac{\partial^2}{\partial z\partial x}\nabla^2K, \quad (9)$$

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t}-\nabla^2\right)v=-Y\frac{\partial\Psi}{\partial z}+\frac{1}{\sigma}J(\Psi,v), \quad (10)$$

$$\left(\frac{\partial}{\partial t}-\nabla^2\right)T=J(\Psi,T)-\frac{\partial\Psi}{\partial x}, \quad (11)$$

$$\left(\frac{\partial}{\partial t}-\tau\nabla^2\right)S=J(\Psi,S)-\frac{\partial\Psi}{\partial x}, \quad (12)$$

and $$\left(\frac{\partial}{\partial t}-\eta'\nabla^2\right)K=HDW+J(\Psi,k), \quad (13)$$

where $\frac{H_z}{\sqrt{h}}=H,\ \tau=\frac{k_S}{k_l},\ \sigma=\frac{v}{k_l},\ R=\frac{g\alpha\Delta Td^3}{vk_l},$

$R_S=\frac{g\beta\Delta Sd^3}{vk_l},\ \eta'=\frac{\eta}{k_l}$ and $D=\frac{d}{dz}$

R and R_S are the thermal and the solutal Rayleigh numbers respectively, σ is the Prandl number, Y^2 is the Taylor number.

Eqs. (9–13) are subjected to the following boundary conditions

$$\Psi=D^2\Psi=T=S=\frac{\partial v}{\partial z}=\frac{\partial K}{\partial z}=0 \quad \text{at } z=0 \text{ and } z=1$$

The linearised equations when the basic state is perturbed by infinitesimal disturbances become

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t}-\nabla^2\right)\nabla^2\Psi=-R\frac{\partial T}{\partial x}+R_S\frac{\partial S}{\partial x}+Y\frac{\partial v}{\partial z}-Q_1^*\frac{\partial^2}{\partial z\partial x}\nabla^2K, \quad (14)$$

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t}-\nabla^2\right)v=-Y\frac{\partial\Psi}{\partial z}, \quad (15)$$

$$\left(\frac{\partial}{\partial t}-\nabla^2\right)T=-\frac{\partial\Psi}{\partial x}, \quad (16)$$

$$\left(\frac{\partial}{\partial t}-\tau\nabla^2\right)S=-\frac{\partial\Psi}{\partial x}, \quad (17)$$

$$\left(\frac{\partial}{\partial t}-\eta'\nabla^2\right)K=-H\frac{\partial^2\Psi}{\partial x\partial z}. \quad (18)$$

Let us assume the solutions of eqs. (14–18) in the form

$$\begin{aligned}\Psi&=Ae^{pt}\sin\pi\alpha x.\sin n\pi z\\S&=Be^{pt}\cos\pi\alpha x.\sin n\pi z,\\v&=Ce^{pt}\sin\pi\alpha x.\cos n\pi z,\\T&=De^{pt}\cos\pi\alpha x.\sin n\pi z,\\K&=Ee^{pt}\cos\pi\alpha x.\cos n\pi z.\end{aligned} \quad (19)$$

Substituting (19) in (14–18) and eliminating A, B, C, D, E, we get for the lowest mode $n=1$,

$$R_{\text{steady}}=\frac{R_S}{\tau}+\frac{\pi^4(1+\alpha^2)^3}{\alpha^2}+\frac{Y^2}{\alpha^2}-\frac{Q\pi^2(1+\alpha^2)}{\alpha\eta'}, \quad (20)$$

where $Q=Q_1^*H$.

For the marginal state being oscillatory $p=ip_1$, p_1 is real we get

$$\begin{aligned}R_{ov}=&\left[vp_1^4(\tau+1+2\sigma+\eta')-p_1^2x^3\right.\\&[(\tau+1)\sigma^2+2\sigma\tau+\eta'\{\tau+\sigma^2+2\sigma(\tau+1)\}]\\&\left.+\sigma^2\tau\eta'x^5\right]/\left[\sigma\alpha^2\pi^2\{\sigma\tau\eta'x^2-p_1^2(\sigma+\tau+\eta')\}\right]\\&+R_S\{\eta'\sigma x^2-p_1^2(\sigma+1+\eta')\}/\\&\{\sigma\tau\eta'x^2-p_1^2(\sigma+\tau+\eta')\}+(\sigma Y^2/\alpha^2)\\&\{\eta'\tau x^2-p_1^2(1+\tau+\eta'\tau)\}/\{\sigma\tau\eta'x^2-p_1^2(\sigma+\tau+\eta')\}\\&-\frac{Q}{\alpha}x\{\sigma\tau x^2-p_1^2(\sigma+\tau+1)\}/\{\sigma\tau\eta'x^2-p_1^2(\sigma+\tau+\eta')\}\end{aligned} \quad (21)$$

where $x=\pi^2(\alpha^2+1)$. (22)

Eq (20) gives R_{steady} if α satisfies

$$\eta'\left[2\pi^4(2\alpha^2-1)(1+\alpha^2)^2-2Y^2\right]+Q\pi^2\alpha(1-\alpha^2)=0 \quad (23)$$

Separating imaginary part of (21) after a bit simplification, we get

$$\begin{aligned}&p_1^4-p_1^2\left[x^2\{\tau(1+\eta')+\sigma^2+\eta'+2\sigma(\eta'+\tau+1)\}\right.\\&\left.+\frac{\sigma}{x}\{R_S\alpha^2\pi^2-R\alpha^2\pi^2+\sigma Y^2\pi^2-Q\alpha\pi^2\}\right]\\&+\left[x^4\{\sigma^2(\tau+\eta'+\eta'\tau)+2\sigma\tau\eta'\}+x\{\alpha^2\pi^2R_S\sigma\right.\\&(\sigma+\eta'+\eta'\sigma)-R\alpha^2\pi^2\sigma(\sigma\tau+\eta'\sigma+\eta'\tau)+\sigma^2Y^2\pi^2\\&\left.(\tau+\eta'+\eta'\tau)-Q\alpha\pi^2\sigma(\sigma\tau+\sigma+\tau)\}\right]=0.\end{aligned} \quad (24)$$

This is a quadratic equation in p_1^2 involving R. It is clear from eq. (24) that if $R<0$, $Q<0$, there are two changes of sign of (24) assuring existence of overstability when the discriminant of this equation is positive.

Elimination of R from (21) and (24) yields an equation of third degree in p_1^2. For the existence of overstability, the parameters be such that it has a real root which must be positive. Solving that cubic equation in p_1^2, we get R_{ov} from (21).

3. Numerical results

Figure 1 depicts α versus $R/10^5$ for various values of η' and τ. It is clear that as τ increases, R_{steady} decreases giving rise to destabilizing effect to the system and it confirms the result obtained by Gupta and Sengupta [2]. From Figure 2, as Y increases R_{steady} increases giving the stabilizing effect. Q has destabilizing effect to the system which is observed from Figure 3. Also the inference that as $|R_S|$ decreases, R_{steady}

increases for the case when salted from below, can be drawn from the same figure. Figure 4 shows that as η' increases, R_c increases and takes its asymptotic value for large η'. For large η', the system is practically unaffected.

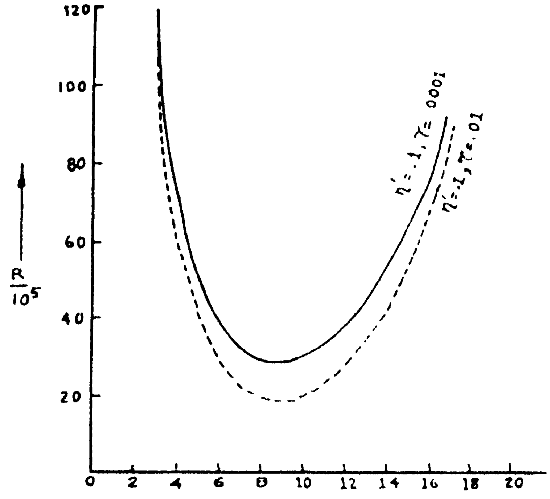

Figure 1. α *versus* $R/10^5$ marginal stability curve when $Y = 10^4$, $R_S = 100$ and $Q = 10$ for different τ

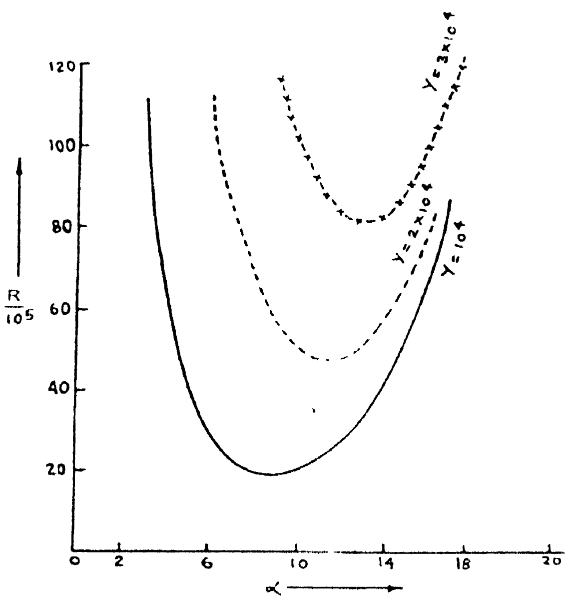

Figure 2. α *versus* $R/10^5$ marginal curve when $R_S = 100$, $\tau = 0.01$, $Q = 10$, $\eta' = 10$ for different Y

It is obvious from Figure 5 that increase of τ decreases R revealing the fact that τ has destabilizing effect to the system as found in the case of stationary convection. Figure 6 ensures the oscillatory behaviour of R, with τ. The attractive decrease of R due to large η' or compared to the marginal state also implies the analogous effect of η' in the instability phenomena which can be visualised from Figure 7. From Figure 8, the effect of Prandl number in the instability mechanism cannot be ignored since this parameter not only changes R_c but also changes the critical wave number to a great extent.

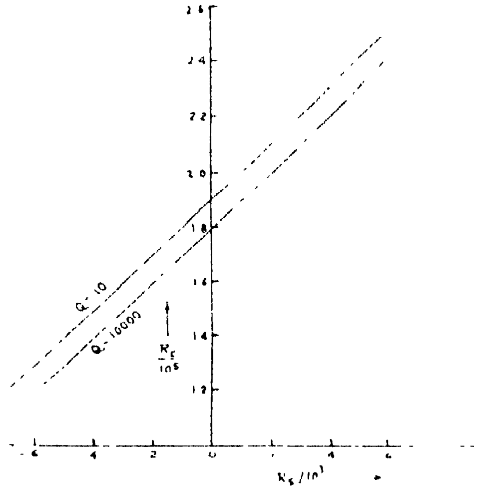

Figure 3. $R_S/10^3$ *versus* $R_c/10^5$ marginal curve when $Y = 10^4$, $\tau = 0.01$, $\eta' = 0.1$ for different Q

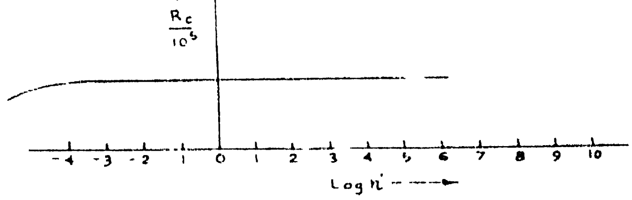

Figure 4. Log η' *versus* $R_c/10^5$ marginal stability curve when $Y = 1000$, $\tau = 0.01$, $Q = 10$, $R_S = 100$

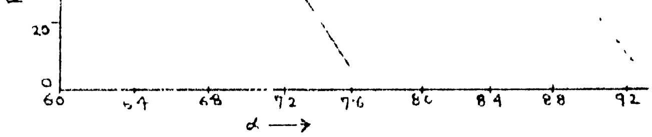

Figure 5. $R/10^5$ *versus* α oscillatory curve when $R_S = 100$, $\sigma = 0.4$, $Q = 10$, $Y = 10^4$, $\eta' = 0.1$ for different τ

4. Finite amplitude steady convection

If disturbances are of finite amplitude and convection being steady, $\frac{\partial}{\partial t} = 0$ in (9–13). Following perturbation method due to Veronis [4], we express all dependent variables in powers of the amplitude ε. So Ψ can be taken as

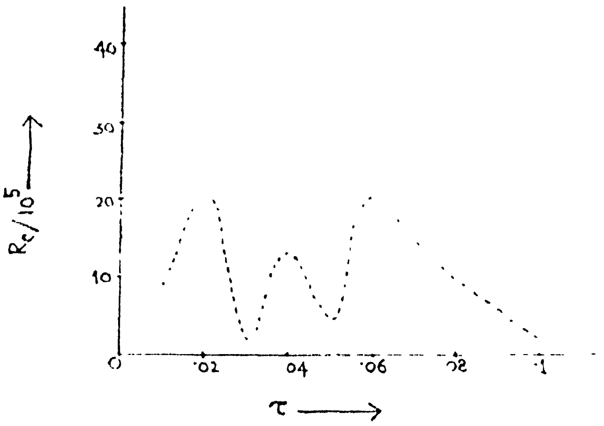

Figure 6. $R_c/10^5$ *versus* τ curve when $R_S = 100$, $\sigma = 0.1$, $Q = 10$, $Y = 10^4$, $\eta' = 0.1$

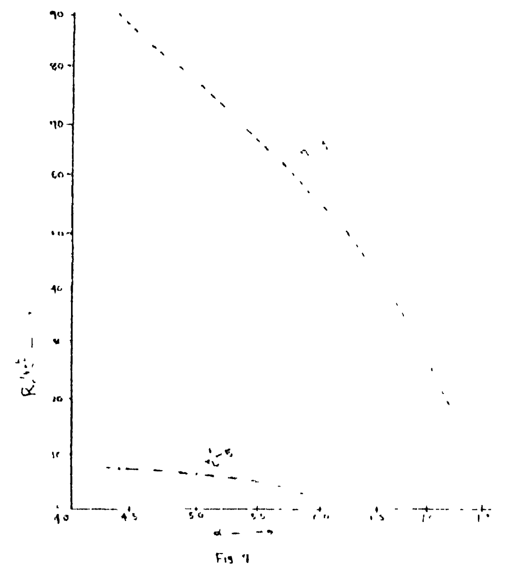

Figure 7. $R/10^5$ *versus* α curve when $R_S = 100$, $\tau = 0.07$, $\sigma = 0.1$, $Q = 10$, $Y = 10^4$ for different η'

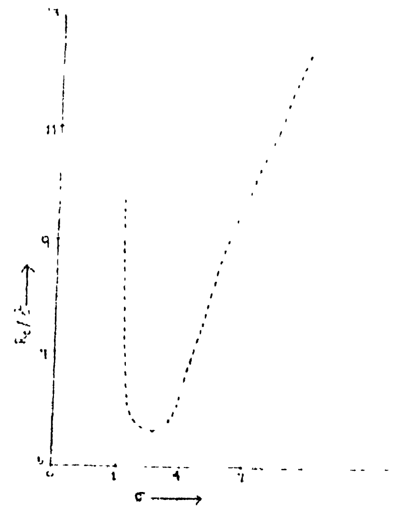

Figure 8. $R_c/10^2$ *versus* σ curve when $R_S = 100$, $\tau = 0.07$, $Q = 10$, $Y = 10^2$, $\eta' = 0.1$

$\psi = \varepsilon\Psi_0 + \varepsilon^2\Psi_1 + \varepsilon^3\Psi_2 + \ldots$ and similarly for other variables

After substitution of the expansions for Ψ, v, T, S, R, K into the governing equations and collecting the coefficients of $\varepsilon, \varepsilon^2, \varepsilon^3$ *etc.*, we eliminate v_0, T_0, S_0, K_0 from the first equation of each set and obtain for Ψ_0

$$L\Psi_0 = -\left[\nabla^6 + Y^2\frac{\partial^2}{\partial z^2} + \left(\frac{R_S}{\tau} - R_0\right)\frac{\partial^2}{\partial x^2} - \frac{Q}{\eta'}\frac{\partial^4}{\partial x^2\partial z^2}\right]\Psi_0 = 0, \tag{25}$$

where $Q = Q_1^* H$.

Solution of (25) under above mentioned boundary conditions for the lowest mode *i.e.* $n = 1$, are

$$\begin{aligned}
\psi_0 &= -\frac{\;}{\pi\alpha}\sin\pi\alpha x.\sin\pi z \\
v_0 &= -\frac{2Y}{\alpha\pi^2(\alpha^2+1)}\sin\pi\alpha x.\cos\pi z \\
T_0 &= -\frac{\;}{\pi^2(\alpha^2+1)}\cos\pi\alpha x.\sin\pi z, \\
S_0 &= -\frac{\;}{\tau\pi^2(\alpha^2+1)}\cos\pi\alpha x.\sin\pi z, \\
K_0 &= -\frac{2H}{\eta'\pi(\alpha^2+1)}\cos\pi\alpha x.\cos\pi z
\end{aligned} \tag{26}$$

Similarly, eliminating v_1, T_1, S_1, K_1 from the second equation of each set and v_2, T_2, S_2, K_2 from the third equation of each set, we derive the equations for Ψ_1 and Ψ_2 as

$$\begin{aligned}
L\Psi_1 = &-R_1\frac{\partial^2\Psi_0}{\partial x^2} - \frac{Y}{\sigma}\frac{\partial}{\partial z}[J(\Psi_0, v_0)] \\
&+R_0\frac{\partial}{\partial x}[J(\Psi_0, T_0)] - \frac{R_S}{\tau}\frac{\partial}{\partial x}[J(\Psi_0, S_0)] \\
&+\frac{1}{\sigma}J(\nabla^2\Psi_0, \nabla^4\Psi_0) + \frac{Q_1^*}{\eta'}\frac{\partial^2}{\partial x\partial z}[J(\Psi_0, k_0)]
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
L\Psi_2 = &-R_2\frac{\partial^2\Psi_0}{\partial x^2} - R_1\frac{\partial^2\Psi_1}{\partial x^2} + R_1\frac{\partial}{\partial x}J(\Psi_0, T_0) \\
&+R_0\frac{\partial}{\partial x}[J(\Psi_0, T_1) + J(\Psi_1, T_0)] \\
&-\frac{Y}{\sigma}\frac{\partial}{\partial z}[J(\Psi_0, v_1) + J(\Psi_1, v_0)] \\
&-\frac{R_S}{\tau}\frac{\partial}{\partial x}[J(\Psi_0, S_1) + J(\Psi_1, S_0)] \\
&+\frac{1}{\sigma}[J(\nabla^2\Psi_0, \nabla^4\Psi_1) + J(\nabla^2\Psi_1, \nabla^4\Psi_0)] \\
&+\frac{Q_1^*}{\eta'}\frac{\partial^2}{\partial x\partial z}[J(\Psi_0, K_1) + J(\Psi_1, K_0)].
\end{aligned} \tag{28}$$

Utilising solutions (26) in (27) we get

$$L\Psi_1 = H^2\alpha^2 R_1\Psi_0.$$

Arguing like Veronis [4], R_1 is calculated so as to cancel the form of Ψ_0 from the right hand side of the above equation because a term of the form Ψ_0 will be a secular term and its presence may hamper the periodicity of the solution. Hence $R_1 = 0$ so that $\mathcal{L}\Psi_1 = 0$. Its solution under the boundary conditions is $\Psi_1 = 0$. Thus one obtains

$$v_1 = \frac{Y}{2\pi^3\alpha^3\sigma(\alpha^2+1)}.\sin 2\pi\alpha x$$

$$T_1 = -\frac{1}{2\pi^3(\alpha^2+1)}.\sin 2\pi z,$$

$$S_1 = -\frac{1}{2\tau^2\pi^3(\alpha^2+1)}.\sin 2\pi z.$$

$$K_1 = \frac{H}{2\eta'^2\pi^2\alpha^2(\alpha^2+1)}.(\cos 2\pi\alpha x - \alpha^2\cos 2\pi z). \tag{29}$$

Utilising all these, we obtain from (28),

$$\mathcal{L}\Psi_2 = \Big[-2\pi\alpha R_2 - \frac{Y^2}{\pi\alpha^3\sigma^2(\alpha^2+1)} - \frac{\alpha R_S}{\pi\tau^3(\alpha^2+1)}$$

$$+\frac{\alpha R_0}{\pi(\alpha^2+1)} - \frac{8Q\pi\alpha}{\eta'^3(1+\alpha^2)}\Big]\sin\pi\alpha x.\sin\pi z$$

$$+\frac{Y^2}{\pi\alpha^3\sigma^2(\alpha^2+1)}\sin 3\pi\alpha x.\sin\pi z$$

$$-\frac{\alpha R_S}{\pi\tau^3(\alpha^2+1)}\ \frac{\alpha R_0}{\pi(\alpha^2+1)}\sin 3\pi z\ \sin\pi\alpha x$$

$$+\frac{12Q\pi\alpha}{\eta'^3(\alpha^2+1)}(\cos^2\pi z + \cos^2\pi\alpha x). \tag{30}$$

Obviously, the first term on the right hand side of (30) has the form of Ψ_0 and hence should vanish. This gives

$$R_2 = \frac{R_0}{2\pi^2(\alpha^2+1)} - \frac{R_S}{2\tau^3\pi^2(\alpha^2+1)}$$
$$-\frac{Y^2}{2\sigma^2\pi^2\alpha^4(\alpha^2+1)} + \frac{4Q}{\eta'^3(1+\alpha^2)}, \tag{31}$$

where R_0 is the critical Rayleigh number at the onset of stationary convection with respect to infinitesimal disturbances. This reveals the fact that the system becomes unstable to finite amplitude steady disturbances before it becomes unstable to disturbances of infinitesimal disturbances. Due to the presence of magnetic field, a reverse effect is taking place leading to the conclusion that η' has destabilising effect to the system.

From (31), it can be remarked that a stable salinity gradient and rotation reinforce each other in causing subcritical instability whereas the magnetic field has reverse behaviour to the instability phenomena.

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